A concept geometry for conceptual spaces

Article in Fuzzy Optimization and Decision Making · October 2006
DOI: 10.1007/s10700-006-0020-1 · Source: DBLP

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A concept geometry for conceptual spaces

John T. Rickard

Abstract This paper generalizes and extends the theory of conceptual spaces as originally proposed by Gardenfors (Conceptual spaces. Cambridge, MA: MIT Press) to provide further geometric representations of both concepts and object observations within a multi-dimensional fuzzy space corresponding to a subset of a unit hypercube. With these representations, we are able directly to calculate normalized scalar measures both of the similarity of two different concepts and of the degree to which an observation satisfies a concept description, and thus to perform inferences with respect to situational assessments and predictions. This capability is directly relevant to the implementation of Levels 2 and 3 data fusion functions using our approach.

Keywords Conceptual spaces · Knowledge representation · Fuzzy sets · Fuzzy systems · Inference mechanisms

1 Introduction

This paper generalizes and extends the theory of conceptual spaces as originally proposed by Gardenfors (2000). In particular, we extend Gardenfors’ geometric approach to object representation to provide further geometric representations of both concepts and object observations within a multi-dimensional fuzzy space corresponding to a subset of a unit hypercube. With these representations, we are able directly to calculate normalized scalar measures both of the similarity
of two different concepts and of the degree to which an observation satisfies a concept description, and thus to perform inferences with respect to situational assessments and predictions. This capability is directly relevant to the implementation of Levels 2 and 3 data fusion functions using our approach.

Aisbett and Gibbon (2001) formulated a general meso level representation by mathematically formalizing the notion of a conceptual space, augmenting the geometric subspace description of Gardenförs with a symbol subspace that associates symbolic labels to concepts and introducing a dynamic mechanism for state transitions between these subspaces as a means of reasoning on external inputs. Their methods can be extended and enriched in combination with our approach, and this will be the subject of a future paper. Raubal (2004) proposed specific metrics for use in conceptual space applications and Zaharia, Leon, and Gâlea (2003) provided algorithms for optimizing categorization in such representations. However, it is fair to say that conceptual spaces have to date received relatively limited attention, even though they provide a novel and very powerful knowledge representation scheme. This paper is more along the lines of Aisbett and Gibbon (2001), i.e., a fundamental extension to the original ideas of Gardenförs (2000).

The organization of this paper is as follows. Section 2 reviews the theory of conceptual spaces, drawing largely upon Gardenförs’ (2000, 2004) work, in some cases nearly verbatim. Section 3 introduces the notion of concepts as labeled graphs, which are then transformed into single points within a unit hypercube space, using Rickard and Yager’s (2006) fuzzy graph representation. Section 4 presents a method for characterizing observations in this same space, and describes a measure for determining the degree to which an observation satisfies a concept, which is essentially a non-linear relative aggregation function. In Sect. 5, we explore applications to Levels 2 and 3 data fusion. Section 6 summarizes the results.

2 Conceptual Spaces

Those who are familiar with conceptual spaces may skip this section and go directly to Sect. 3.

2.1 Motivation

The original motivation for conceptual spaces was to provide an eclectic approach to knowledge representation that exploits the geometric structures embodied in many of the attributes of the real world, while preserving a semantically meaningful description of these attributes. Thus conceptual spaces lie midway between the two dominant modalities for knowledge representation, namely the symbolic and associationist approaches.

The symbolic approach employs symbolic representations of knowledge and propositional statements (i.e., rules) to capture the relationships between symbols, and thus reduces cognition to computation in the form of symbol
manipulation. The associationist approach typically deals with much lower level representations of knowledge (often in the form of raw data or low level features with little if any semantic significance), and relies upon associations between these data elements for performing cognitive tasks. Its most common embodiment is in connectionist (i.e., neural network) architectures, where the association of data to individual neuronal elements determines their degree of activation, and these activations in turn determine the overall network state.

While both of these approaches have their strengths, there lies a considerable gulf between them that leaves unaddressed many important aspects of cognition. To quote Gardenfors (2000, p. 1),

“There are aspects of cognitive phenomena, however, for which neither symbolic representation nor associationism appear to offer appropriate modeling tools. In particular it appears that mechanisms of concept acquisition, which are paramount for the understanding of many cognitive phenomena, cannot be given a satisfactory treatment in any of these representational forms. Concept learning is closely tied to the notion of similarity, which has turned out to be problematic for the symbolic and associationistic approaches.”

Conceptual spaces utilize a geometric representation of knowledge that enables similarity to be modeled and computed in a natural way, using appropriate metrics.

2.2 Dimensions

The starting point for a conceptual space is a set of dimensions capable of describing the quality attributes of the information to be represented. These dimensions can be either psychophysical (which measure human phenomenal responses and are thus semantically meaningful within the constructs of natural language) or scientific (which measure the values associated with sensors, actuators, etc.) For a given application, there is generally no unique assignment of dimensions. Instead, those who are most familiar with the application (often referred to as “subject matter experts” or SMEs) will specify the appropriate dimensions that capture its essential qualities. This represents the bare minimum of “knowledge engineering” required to model a problem.

Dimensions generally possess geometric and/or topological structures that enable us to measure distances between two values. In the case of continuous linear variables such as time or spatial co-ordinate axes, the distance is typically a function of the difference between the two values. However, many variables have more complex distance functions, e.g., where the underlying quality has a circular structure, in which case the distance might be specified as the length of an arc generated by convex combinations in both the radius and angle dimensions, using the same parameter to trace the arc. We can also envision discrete structures such as graphs, where the distance between two nodes is measured by the shortest number of links connecting them.
2.3 Domains

The above dimensions are organized into multiple domains. These domains need not be Cartesian co-ordinate systems, as illustrated in Fig. 1, which illustrates three different domain co-ordinate structures.

The defining feature of a domain is that its dimensions are integral, in the sense that a logical distance measure incorporating all of the domain dimensions can be assigned within the domain, whereas no such measure can be assigned across domains. For example, in the conceptual space of apples, the color domain can be represented by the three dimensions of the color spindle (the first domain in Fig. 1), and the taste domain can be represented by the surface of a tetrahedron with vertices sweet, sour, bitter and salty (the second domain in Fig. 1). One can logically assign a distance measure between two colors or between two tastes, but not between a color and a taste. Thus the color and taste dimensions belong in separate domains.

In the conceptual spaces familiar to humans, some domains are innate and apparently hard-wired in our sensory apparatus, e.g., color, aural pitch and ordinary three-dimensional space. Other domains are learned or culturally dependent, while still others are induced by science. The ability to add domains to a conceptual space further to define information regarding objects of interest endows it with virtually unlimited capability for knowledge representation.

Objects in a conceptual space are represented by points, in each domain, that characterize their dimensional values. This constitutes the fundamental geometric character of conceptual spaces for knowledge representation. Within each domain, we can measure the distance between two objects as the distance between their corresponding points.

Inherent to the notion of distance is the notion of similarity. Two values are considered to be similar to the extent that the distance between them is small, and vice versa. While the distance \( d(x, y) \) between two points \( x \) and \( y \) may be measured directly using, for example, the Euclidean distance metric, we often elect to describe similarity as a fuzzy function of distance \( s(x, y) = f(d(x, y)) \) taking values in the interval [0,1], which expresses the fuzzy degree to which the two points \( x \) and \( y \) are similar. If \( d(x, y) \leq \delta \) for some \( \delta \geq 0 \), the similarity

![Fig. 1 Three examples of conceptual space domains](image)
$s(x,y)$ is considered to be unity, while if $d(x,y) \geq \gamma$ for some $\gamma > 0$, $s(x,y)$ is considered to be zero. For $\delta < d(x,y) < \gamma$, $s(x,y)$ takes on values between zero and unity, declining monotonically with increasing $d(x,y)$. By working with fuzzy values for similarity, we can take advantage of many concepts from fuzzy set theory that provide a linguistic interpretation of results, while preserving the essential geometric aspects of our knowledge representation.

2.4 Properties

A property is a convex region in some domain. The notion of convexity for property regions arises from the logical assumption that if two objects $v_1$ and $v_2$ possess some property, then objects located between $v_1$ and $v_2$ should likewise possess that property. The “betweenness” in some domains may not correspond to “lying on a line between”, e.g., in a circular cross-section of the color domain in Fig. 1, we use a joint convex combination of radius and angle co-ordinates to define an arc whose locus of points is “between” $v_1$ and $v_2$. In natural languages, properties often correspond to adjective-like descriptions (e.g., “red”, “tall”, or “round”) in a particular domain. Properties can also capture more complex descriptions of objects, including shapes, actions and functional characteristics, as described in Gardenförs (2000, pp. 94–99). They can also be defined in probabilistic or fuzzy terms, which provide a distinct advantage over representational schemes that require strict set membership.

The notion of properties is related to prototype theory, where certain members of a category of objects are considered to be most representative (i.e., prototypes) of the category as a whole, and to clustering theory, where the centroid of a cluster is taken to be representative of the cluster members. Given a set of prototypes or cluster centroids within a domain, we can use the distance metric of the domain to create a Voronoi tessellation of the domain, which divides the total volume into regions such that all points within a given region are closer to the prototype or cluster centroid contained in that region than they are to any other prototype or cluster centroid. This notion can be generalized by considering prototypical circular or elliptical volumes rather than points, and computing Voronoi tessellations with respect to the boundaries of these volumes, which allows both the central tendency and the dispersion of the exemplars of a particular property to be taken into account in partitioning a domain into property regions.

Objects whose co-ordinates lie within one of these tessellation regions are associated to the corresponding property. More generally, using the similarity measure associated with a domain, we can determine the degree to which the qualities of a particular object relative to a given domain satisfy different properties within that domain by calculating their similarity to the prototype or centroid of the property region. This will prove quite useful in our approach to matching observations to concepts in Sect. 5.
2.5 Concepts

A concept is a combination of properties, typically across multiple domains, along with the salience weights associated with each property and the correlations (used in the sense of co-occurrences, as opposed to statistical correlations which can be positive or negative) between properties. The choice of properties is predicated upon the descriptive features of the application. The salience weights may be dependent upon the context. For example, the concept “apple” may include the domains of color, taste, surface texture, shape, nutritional content and density, with multiple property regions within each domain to account for the various types of apples. In a display context, color properties may be quite important, while in a cooking context, much less so. Furthermore, there will be distinct correlations between red, green or yellow color properties and a smooth surface texture property, and likewise between brown color and a wrinkled surface texture, and there will be anti-correlations between the converse pairings, e.g., brown with smooth, which do not occur together.

The distinction between concepts and properties is often obliterated in other representational schemes, which lump properties into predicates, classes or low-level features. Concepts typically describe noun- or verb-like (when time is included as a dimension) objects. On the surface, concepts bear some resemblance to the popular representational scheme of frames with slots for different features as originally proposed by Minsky (1975). However, as we shall see in Sects. 3 and 4, the geometric structure of concepts within the conceptual space framework enables such useful operations as calculating the similarity of observations to concepts, of concepts to other concepts, and even such cognitively sophisticated operations as the creation of metaphors.

2.6 Concept combinations

Cognition frequently requires the combination of concepts. The traditional symbolic approach to concept combination is through conjunctions of predicates, which is equivalent to the intersection of classes. However, many concepts cannot be adequately modeled by conjunctions. For example, white Zinfandel is not a white wine, nor for that matter are white wines actually white, but rather a yellow color.

The conceptual space framework admits more flexible combinations of concepts. In addition to the intersections of property regions within domains (equivalent to conjunction), some possible concept combinations include: replacement of properties (e.g., the concept pink elephant replaces the gray color property of elephants with a pink property, retaining all other properties); blocking of domains (e.g., the concept stone lion block all domains associated with “lion” except for the shape domain); and contrast classes. In the latter instance, the combination of concepts $X$ and $Y$ is determined by replacing the property regions of $Y$ with the corresponding regions of $X$ that are confined to the contrast class defined by $Y$. This allows us to distinguish the modifiers in such
combined concepts as red book, red wine, red hair, red skin, red soil and redwood, in all of whose cases the precise color region for the property “red” is predicated upon the object \(Y\).

3 Concept representations

This section begins the new material in this paper. We first describe how concepts can be represented graphically, and then be transformed to a point in a unit hypercube that constitutes a multi-dimensional fuzzy set. We then present a method for analytical similarity comparisons between concepts using these point representations.

3.1 Graphical representations

We recall from Section 2.5 that a concept is a combination of properties, typically across multiple domains, along with the salience weights associated with each property and the correlations between properties. We can capture the elements of a concept in an attributed graph \(C\), where each node in the graph corresponds to a property \(P_i\) with its corresponding salience weight \(w_i\) as an attribute.

The weighted edges between nodes are determined by the domain structure of the concept. We assign directional edges between all pairs of nodes corresponding to properties in \textit{different} domains. Associated with each such edge is a directional edge strength \(C_{ij}\) which, in the case of crisp property memberships, equals the fraction of an ensemble of object exemplars for the concept having property \(P_i\) that also have property \(P_j\). \(C_{ij} = 1\) if and only if all objects having property \(P_i\) in one domain also have property \(P_j\) in another, but in general \(C_{ij} \neq C_{ji}, i \neq j\). We set \(C_{ij} = 0\) if the pair \((i, j)\) corresponds to disjoint properties within the same domain; otherwise, we treat these values the same as for cross-domain properties. Our approach also accommodates fuzzy property membership values and the resulting correlations. Whether crisp or fuzzy, these correlations can be learned straightforwardly as described below, given an exemplar data set, or in its absence they can be specified a priori.

For example, if all red (\(P_r\)), green (\(P_g\)) or yellow (\(P_y\)) apples are smooth textured (\(P_s\)), then \(C_{rs} = C_{gs} = C_{ys} = 1\). On the other hand, \(C_{sr}, C_{sg}, C_{sy}\) are all less than unity and one can consider their values to be analogous to the conditional probabilities that a smooth textured apple will have the corresponding color. Likewise, if all brown (\(P_b\)) apples are of wrinkled texture (\(P_w\)) and vice versa, then we have \(C_{bw} = C_{wb} = 1\). Conversely, the anti-correlations correspond to zero edge weights, so that \(C_{rw} = C_{gw} = C_{yw} = C_{wr} = C_{wg} = C_{wy} = C_{bs} = C_{sb} = 0\).

Below we show an example “apple” concept graph connection matrix, where we have placed zeroes in all entries corresponding to pairs of properties in the same domain as described above, since these are excluded from the notion of a concept.
Note that concept connection matrices are square, but generally non-symmetric.

Concept matrices can be learned from a training data set. Suppose we are given a set of $N$ property vectors $P_j$ in multiple domains corresponding to a concept $C$, a set of parameterized similarity measures $s_k(x, y; \theta)$ parameterized by $\theta$, and a training set of observations $z_{ik}$ representing concept $C$. Let $z_{Pj}$ denote the point or region corresponding to property $P_j$ in $C$. For each training observation, we calculate its vector of similarities $q_i$ to the properties $P_j$ in concept $C$:

$$q_i = \begin{bmatrix} q_{i1} & q_{i2} & \cdots & q_{iN} \end{bmatrix}^T = \begin{bmatrix} s_1(z_{i1}, z_{P1}) & s_2(z_{i2}, z_{P2}) & \cdots & s_N(z_{iN}, z_{PN}) \end{bmatrix}^T. \quad (2)$$

Thus the $j$th element of this vector represents the similarity of the $i$th training observation to the $j$th property in $C$. We then calculate the concept matrix for concept $C$ as

$$C_{jk} = \sum_i \min \left( \frac{q_{ij}, q_{ik}}{q_{ij}} \right), \quad (3)$$

where the indices $(j, k)$ correspond to properties in separate domains.

In the case of crisp property membership values, i.e., $q_{ij} = 0$ or 1, this expression corresponds to our previous definition as the fraction of training observations having property $j$ that also have property $k$. For fuzzy degrees of property membership $0 \leq q_{ij} \leq 1$, this expression represents the average ratio of the observations’ fuzzy membership in both properties (as specified by the min function) to their fuzzy membership in the first property. As we shall describe more fully in Sect. 4.2, this expression can also be viewed as the empirical fuzzy subsethood of the training set’s membership in the $j$th property with respect to its membership in the $k$th property.

In constructing a decision system that must decide between different concepts based upon observations (which is the focus of this paper), it is often possible to specify an objective function for measuring the efficacy of the similarity measures and concept matrices resulting from a particular choice of parameters.
θ over the training data set. One can then use an evolutionary optimization algorithm over the parameters θ in the similarity measures to determine an optimal set of parameters for maximizing this objective function. This will be discussed in future papers for particular applications.

3.2 Concepts as points

We now employ Rickard and Yager’s (2006) hypercube graph representation to transform the connection matrix C into a point in the $N^2$-dimensional unit hypercube, where $N$ is the matrix dimension. This is accomplished by “unwinding” the matrix rows into a concept vector $c$, where each element of the vector corresponds to the correlation strength between a pair of properties, as illustrated in Fig. 2.

The indexing convention we employ to relate the elements $c_k$ of the vector $c$ with the elements $C_{i,j}$ of the matrix $C$ is as follows:

$$
\begin{align*}
c_1 & \leftrightarrow C_{1,1} \\
\vdots & \\
c_N & \leftrightarrow C_{1,N} \\
c_{N+1} & \leftrightarrow C_{2,1} \\
\vdots & \\
c_{2N} & \leftrightarrow C_{2,N} \\
\vdots & \\
c_{N^2} & \leftrightarrow C_{N,N}
\end{align*}
$$

or, relating the index $k$ to the row, column pair $(i, j)$ of $C$,

$$c_k = C_{(i-1)N+j}, \quad i = 1, \ldots, N, \quad j = 1, \ldots, N. \quad (5)$$

Thus the graph represented by the matrix $C$ now becomes a point $c$ in the $N^2$-dimensional hypercube, analogous to a fuzzy set with corresponding membership co-ordinates in each of the latter dimensions. This hypercube description allows us to invoke fuzzy set theoretic concepts for graph representation and characterization.

Fig. 2 Representation of a connection matrix as a point in the unit hypercube.
The salience weight $\omega_k$ of each property pair can be taken, e.g., as the product of the individual property salience weights: $\omega_k = w_i w_j, k \leftrightarrow (i, j)$, where the vector index $k$ corresponds to the property pair $(i, j)$ as above (other weighting combinations for the salience weights such as the minimum, or any number of different averaging operators, are also possible.)

Thus we have translated Gardenfors’ definition of a concept into a point in the unit hypercube, which further embellishes the geometric structure of conceptual spaces, and more importantly, enables us to treat concepts as multi-dimensional fuzzy sets in the space of property pairs. Along with the point $c$, we have a vector $\omega$ of salience weights associated with the property pairs.

### 3.3 Concept similarity

As described by Rickard and Yager (2006), this representation enables us immediately to compare concept similarities analytically using the fuzzy mutual subsethood measure (Kosko, 1997). Mutual subsethood is the fundamental similarity measure for fuzzy sets, and is defined for a pair of concepts $a$ and $b$ as

$$E(a, b) = \frac{\sum_k \min(a_k, b_k)}{\sum_k \max(a_k, b_k)}.$$  \hspace{1cm} (6)

This symmetric measure has the properties $0 \leq E(a, b) \leq 1$, $E(a, b) = 1 \iff a = b$ and $E(a, b) = 0$ if and only if for every $k, a_k \neq 0 \Rightarrow b_k = 0$ and $b_k \neq 0 \Rightarrow a_k = 0$. In the degenerate case where $a_k = b_k = 0$ for all $k$, we define $E(a, b) = 0$.

Figure 3 illustrates mutual subsethood geometrically for a two-dimensional case as the ratio of the Hamming norms \textit{(not} the Euclidean norms) of two fuzzy sets derived from $a$ and $b$, namely their intersections and unions, respectively.

The mutual subsethood measure can incorporate dimensional salience weights $0 \leq \omega_k \leq 1$ in straightforward fashion. We define the salience-weighted mutual subsethood $E_\omega(a, b)$ with respect to the weight vector $\omega$, by

$$E_\omega(a, b) \triangleq \frac{\sum_k \omega_k \min(a_k, b_k)}{\sum_k \omega_k \max(a_k, b_k)}.$$ \hspace{1cm} (7)

Note that $E_\omega(a, b)$ satisfies the same properties mentioned above as $E(a, b)$, but it re-apportions the relative contributions of the various dimensions to the ratio in proportion to $\omega_k$. Equipped with this measure, we can compute the \textit{normalized, salience-weighted concept similarity} between any two concepts.

In cases where the two concepts do not share all properties (or even domains) in common, we take the union of the property sets for the computation of augmented connection matrices for each concept, where for a property $P_i$ in the union set not held by one of the concepts, all corresponding entries for the $i$th row and column in that concepts’ connection matrix are zero, including the diagonal term. These non-common nodes contribute nothing to the numerator of (6) or (7) since either $a_k = 0$ or $b_k = 0$ for all $k$ corresponding to property pairings.
(P_i, P_j) in the augmented connection matrices where one of the properties is not held by one of the concepts. However, they do increase the denominator of these expressions since one of the terms a_k or b_k for the k corresponding to the property pairing (P_i, P_j) will always be non-zero, thereby reducing the overall similarity between concepts in proportion to the connection strengths (salience-weighted in the case of (7)) of properties that are not shared, whatever may be the similarities between connection strengths of properties held in common.

We further note that the above concept representation and similarity measure applies equally well to the various forms of concept combinations described in Section 2.6, since the result of all such combinations retains the basic structure of a concept as a combination of properties with associated correlations and salience weights.

The ability to represent concepts geometrically and to calculate their similarity as described above can be used in at least two different ways for cognitive processing. If the sets of properties involved between two concepts have some overlap (which might be the case for two concepts operating in the same application), then the mutual subsethood between the concepts provides a measure of the degree of “orthogonality” between concepts, somewhat analogous to the normalized inner product of two vectors, where a small mutual subsethood implies nearly orthogonal concepts. Note however, that the mutual subsethood measure takes into account not only the directions, but also the magnitudes of the two hypercube points representing the concepts. This measure can be used to identify redundancy in a library of concepts.

Additionally, we can use clustering processing such as described by Rickard, Yager, and Miller (2005) on a library of concepts to identify regions in “concept space” that are “semantically denser” than other regions, which provides clues to the meta-structure of our knowledge of a particular application.
Alternatively, in cases where the properties involved in two concepts have no overlap (as might be the case if they lie in different applications) we can construct *metaphors* between the two applications by starting with a concept in one application, associating the involved properties in this application with analogous properties in the second application, and then applying the same concept matrix to the latter properties to create the metaphor.

4 Concept/observation similarity

Ultimately, we wish to compare observations about the real world to a library of concepts stored in our cognitive systems in order to perform inferences upon these observations. A concept is generally more complex than an observation, since concepts encode the correlations among all possible combinations of properties between domains, whereas a specific observation corresponds to a single point in each domain. Thus a concept describes the ensemble properties of all objects that are representative of the concept, while an observation is a single realization corresponding to a particular object. For example, a particular apple typically possesses only one color property and surface texture, while the concept “apple” involves all possible colors and surface textures, and their correlations, for all apples. In order to calculate the similarity of an observation to a concept, we must cast the former in the same geometric space as the latter, and then apply an appropriate measure for this purpose. These topics are addressed in this section.

4.1 Object correlation representation

The starting point for relating an observation \( x \) (i.e., a set of vector points, one in each domain of the conceptual space) to a concept is to calculate its similarity \( s_i = s(x, P_i) \) \((0 \leq s_i \leq 1)\) to each property \( P_i \) involved in the concept, using the similarity measure derived from the distance metric associated with each domain. Typically this involves calculating the distance between \( x \) and the centroid (i.e., prototype point) of the property region \( P_i \) and then mapping this distance value into the normalized similarity measure as described in Sect. 2.3.

In a given domain \( D_n \) of a conceptual space containing multiple properties \( P_j, j \in D_n \), a particular observation \( x \) will have various similarities \( s_j, j \in D_n \) to the properties \( P_k \) within that domain. Between two different domains, the minimum function \( \min(s_j, s_k) \) describes the fuzzy degree to which a *pair* of properties is possessed by the observation \( x \). We require a means of representing this information in the same geometric space we are using to describe concepts. Note, however, that concepts generally do not involve all possible pairings of properties between domains. Thus we desire to capture the information in our observation \( x \) that is relevant to a given concept \( C \). To this end, let \( I_C \) denote the indicator set of the properties and property pairings involved in the concept.
C, i.e., \( C_{ij} > 0 \) for all \((i,j) \in I_C\) and \( C_{ij} = 0 \) for all \((i,j) \notin I_C\). We then define

\[
\Phi_{nm}(C) \triangleq \max_{j \in D_n, k \in D_m} \min_{(j,k) \in I_C} (s_j, s_k)
\]

as the largest of the minima of the pairwise similarity values between pairs of properties in two domains, with

\[
\Phi_{nn}(C) \triangleq \max_{j \in D_n, j \in I_C} s_j
\]

Clearly we can use other pairwise aggregation functions in (8) besides the minimum operator, such as products, ordered weighted averages (e.g., Yager & Kacprzyk, 1997) and Minkowski means, but the minimum function is both conservative and computationally attractive.

Thus \( \Phi_{nn}(C) \) is the largest intra-domain similarity of observation \( x \) to a property associated with \( C \) in domain \( D_n \), while \( \Phi_{nm}(C) \) is the largest inter-domain similarity of observation \( x \) to a pairs of properties between domains \( D_n \) and \( D_m \) associated with \( C \), where the individual pairwise similarity is taken as the minimum similarity of the pair. \( \Phi \) is a symmetric, \( M \)-dimensional square matrix that captures the best match of observation \( x \) to individual properties within domains and to pairs of properties across domains associated with \( C \), where \( M \) is the number of domains involved in the concept.

We now specify a “connection” matrix \( \Psi(C) \) for the observation \( x \) that is suitable for similarity calculations with respect to concept connection matrices as follows:

\[
\Psi_{ij}(C) = \begin{cases} 
\Phi_{nm}(C) & \text{for all } i \in D_n, j \in D_m, (i,j) \in I_C \\
0 & \text{otherwise}
\end{cases}
\]

where \( i \) ranges over all indices assigned to properties in \( D_n \) and \( j \) ranges over all indices assigned to properties in \( D_m \) that are associated with \( C \), using the same indexing as with \( C \). Provided that these indices are assigned in consecutive groups to the properties of each domain, the matrix \( \Psi(C) \) will have a block-rectangular structure in which all elements involving allowable property matches between a given pair of domains (i.e., contained in \( I_C \)) will have identical values, equal to the largest allowable pairwise property similarity that exists between the pair of domains. Thus for every allowable pair of properties between two given domains, we credit to all such pairs the highest property pair similarity score for the object between these two domains.

For example, suppose a particular observed apple has similarity values with respect to the color properties red, green, yellow and brown given by 0.9, 0.1, 0.3, and 0, respectively, and has similarity values with respect to the smooth and wrinkled surface texture properties given by 0.7 and 0.2, respectively. This object’s connection matrix \( \Psi \) is then given by
where the internal block-rectangular structure of the matrix has been highlighted. (Note that in general, the object similarities do not sum to unity in any given domain.)

Since the matrix $\Psi(C)$ is of the same dimension as the concept connection matrix $C$, we can perform the same transformation described in Sect 3.2 to covert $\Psi(C)$ into a vector $\psi(C)$ in the unit hypercube. The usefulness of this representational scheme will become apparent when we describe the measure to be used for concept/observation similarity. This measure is based upon the notion of fuzzy subsethood (Kosko, 1992), which we describe mathematically in the next section.

4.2 Fuzzy subsethood

As distinct from traditional set theory, every fuzzy set is a subset (to a quantifiable fuzzy degree) of every other fuzzy set. The basic measure of the degree to which fuzzy set $A$ is a subset of fuzzy set $B$ is fuzzy subsethood, defined as:

$$S(A, B) = 1 - \frac{M(A \cap B^*)}{M(A)} = 1 - \frac{\sum \max (0, A_i - B_i^*)}{\sum A_i} = \frac{\sum \min (A_i, B_i)}{\sum A_i},$$

where $M(A \cap B^*)$ is the Hamming distance between $A$ and $B^*$, the nearest point to $A$ contained within $F(2^B)$, the fuzzy power set of $B$ (i.e., $F(2^B)$ is the set of all proper subsets sets of $B$), and $M(A)$ is the Hamming norm of fuzzy set $A$. From (12) we have $0 \leq S(A, B) \leq 1$, $S(A, B) = 0 \Rightarrow B = \Phi$, the null fuzzy set, $S(A, B) = 1 \Rightarrow A \subset B$ ($A$ is a proper subset of $B$), and $0 \leq S(A, B) < 1$ implies that $A$ lies outside of $F(2^B)$. Note that fuzzy subsethood in general is not symmetric, i.e., $S(A, B) \neq S(B, A)$.

Figure 4 illustrates, for three different fuzzy sets $A_1, A_2$ and $A_3$, their corresponding nearest fuzzy sets $B_1^*, B_2^*$ and $B_3^*$ in the fuzzy power set $F(2^B)$ of a fourth fuzzy set $B$. The Hamming distances between these respective sets represents the numerator term in (12).

For example, if fuzzy set $A_3$ has components $\{\frac{5}{8}, \frac{3}{8}\}$ and $B$ has components $\{\frac{1}{4}, \frac{3}{4}\}$, then $M(A_3 \cap B^*) = \frac{3}{8}$, and $M(A_3) = 1$, so $S(A_3, B) = \frac{5}{8}$. 

\[\begin{array}{cccccc}
\text{Red} & \text{Green} & \text{Yellow} & \text{Brown} & \text{Smooth} & \text{Wrinkled} \\
0.9 & 0 & 0 & 0 & 0.7 & 0 \\
0 & 0.9 & 0 & 0 & 0.7 & 0 \\
0 & 0 & 0.9 & 0 & 0.7 & 0 \\
0 & 0 & 0 & 0.9 & 0 & 0.7 \\
0.7 & 0.7 & 0.7 & 0 & 0.7 & 0 \\
0 & 0 & 0 & 0.7 & 0 & 0.7 \\
\end{array}\]
The fundamental significance of subsethood derives from the subsethood theorem (Kosko, 1992):

$$S(A, B) = \frac{M(A \cap B)}{M(A)}.$$  \hspace{1cm} (13)

As was the case with mutual subsethood, we can employ dimensional salience weights $0 \leq \omega_k \leq 1$ on the components of $A$ and $B$ to calculate a weighted fuzzy subsethood:

$$S_\omega (A, B) = \frac{M_\omega(A \cap B)}{M_\omega (A)} = \frac{\sum_k \omega_k \min(A_k, B_k)}{\sum_k \omega_k A_k}.$$  \hspace{1cm} (14)

Analogous to (7), these weights re-apportion the relative contributions of the dimensional terms in the numerator and denominator of this expression in proportion to $\omega_k$.

The fuzzy subsethood theorem also leads immediately to the Bayesian-like identity

$$S(A, B) = \frac{S(B, A) M(B)}{M(A)}.$$  \hspace{1cm} (15)

The relationship between fuzzy set theory and empirical probability theory becomes apparent by letting $X$ be the point $\{1, \ldots, 1\}$ in $I^n$, i.e., the outer vertex of the unit hypercube, and let $a_i$ be the binary indicator function of an event outcome in the $i$th trial of a random experiment (e.g., the event of heads in an arbitrarily biased coin toss) repeated $n$ times. Thus the fuzzy power set $2^X$ of $X$ includes the “universe of discourse” (i.e., the set of all possible outcomes, which correspond to the vertices of the unit hypercube) for the entire experiment, and
\[ S(X, A) = \frac{M(A \cap X)}{M(X)} = \frac{M(A)}{M(X)} = \frac{n_A}{n}. \]  

(16)

where \( n_A \) denotes the number of successful outcomes of the event in question. In other words, the subsethood of \( X \) in one of its binary component subsets (corresponding to one of the vertices of the unit hypercube) is simply the relative frequency of occurrence of the event in question.

Thus probability (in both its Bayesian and relative frequency interpretations) is directly related to subsethood. The above illustrates the “counting” aspect of fuzzy subsethood when applied to crisp outcomes, which also is central to probability theory (the Borel field over which a probability space is defined is by definition a sigma-field of events, containing all atomic events and their countable combinations). However, note that (12) admits a “partial count” term in both the numerator and denominator when the fuzzy sets in question do not reside at a vertex of \( I^n \).

In the context of our graph representation described in Section 3.2, fuzzy subsethood measures the degree to which graph \( A \) is a subset of graph \( B \), using their corresponding hypercube vector representations \( a \) and \( b \). If \( a \) is contained in the fuzzy power set \( F(2^b) \) of \( b \), then \( S(a, b) = 1 \), which is equivalent to the statement that, for every edge between pairs of vertices in graph \( A \), there exists a corresponding edge in graph \( B \) with equal or greater connection strength. If \( S(a, b) = 0 \), then \( A \) and \( B \) have no edges in common between corresponding pairs of vertices. Finally, if \( 0 < S(a, b) < 1 \), then \( A \) and \( B \) have some (non-zero strength) edges in common, but there is no strict subordination of edge connection strengths between them, i.e., \( a \cap b \neq a \cup b \).

4.3 Concept/observation similarity measure

With the above mathematical apparatus, we can now define the similarity \( \sigma(C, x) \) between an observation \( x \) and a concept \( C \) to be the fuzzy subsethood of the corresponding hypercube concept vector \( c \) in the hypercube observation vector \( \psi(C) \) constructed as in Sect. 4.1:

\[
\sigma(C, x) = S(c, \psi(C)) = \frac{\sum_{i \in I_C} \min(c_i, \psi_i(C))}{\sum_{i \in I_C} c_i}. \]

(17)

Intuitively, this measures the degree to which an observation’s pairwise property similarities in the context of \( C \), as defined by (8) and (10), equal or exceed those required by the concept. This measure is not only mathematically well-founded, but is computationally appealing as well.

As an example of this calculation, consider the two matrices described above for the concept “apple” in (1) and the observation of a particular apple in (11):
After transforming these matrices into fuzzy vectors $c$ and $\psi$, respectively, as described previously, we use (17) to calculate the similarity of this particular apple to the concept “apple”:

$$S(c, \psi) = \frac{\sum_{i=1}^{N} \min(c_i, \psi_i)}{\sum_{i=1}^{N} c_i} = \frac{4 \times 0.9 + 7 \times 0.7 + 0.5 + 0.2 + 0.3}{11 \times 1 + 0.5 + 0.2 + 0.3} = 0.7917.$$  

Thus this object’s overall fit to the concept “apple” of 0.7917 lies roughly midway between its best fit in the color domain (0.9 to the color “red”) and its best fit in the surface texture domain (0.7 to the texture “smooth”).

In summary, this section has developed a rationale and approach for representing an observation in the same unit hypercube space as the concept representations developed in Sect. 3, along with an analytical similarity measure (fuzzy subsethood) for determining the match of this observation to a concept. Given a library of concepts, one can thus compare an observation to members of this library (making use of cluster-tree based search strategies (Yu & Zhang, 2003) if desired to reduce the number of comparisons needed) and compile a ranked list of concepts in order of their similarity to the observation. Given the inherently normalized aspect of the fuzzy subsethood measure, these similarity values can be thresholded to identify concepts to which the observation has adequate similarity. If an observation proves dissimilar (i.e., below threshold) to all concepts in the library, it is classified as a novelty, which warrants further investigation and/or suggests the need for additional concepts.

This approach forms the basis for automated situational assessment in Level 2 data fusion and automated prediction in Level 3 fusion. The following section addresses these applications.

### 5 Applications to levels 2 and 3 fusion

Level 2 fusion is concerned with situation assessment, i.e., the group characterization of a set of objects derived from Level 1 fusion. Level 3 fusion is concerned with threat assessment, i.e., the prediction of an adversary’s intentions, given...
their observed group character. In both instances, the fusion system implementations for specific application areas often embody ad hoc, symbolic-based approaches that lack any capacity for generalization to other applications.

5.1 Level 2 fusion

Stubberud, Shea, and Klamer (2003a) have pointed out the need for a state-based representation of Level 2 objects as a first step towards establishing a common architecture for Level 2 fusion, such as exists for Level 1 fusion. The starting point for this representation is to perform the detection of candidate Level 2 objects via the clustering of Level 1 objects. These authors (2003b) also proposed certain metrics for use in the association of these Level 2 objects to templates in order to provide a symbolic characterization of these objects. These metrics include a normalized $\chi^2$ residual for kinematic distance, cardinality and gap metrics for group compositional differences, a Hausdorff-type metric for group separation, an invariant moment metric for group formation differences, and an area metric for group region of influence.

We suggest here that the conceptual spaces approach outlined in this paper provides precisely the type of general architecture sought after in Stubberud, Shea, and Klamer (2003a), but in a richer descriptive context than would be offered by a symbolic representation. The different vector components of the Level 2 state prescribed in this work reside in individual domains of the conceptual space representing an application. The metrics proposed in Stubberud, Shea, and Klamer (2003b) are associated with these domains. Additional domains and metrics may be added as needed further to characterize Level 2 objects. Rather than describing different regions of the state space symbolically and using rule-based associations of Level 2 objects to templates, we employ the more powerful geometric descriptions of properties and concepts inherent in the conceptual space paradigm. Finally, we supply the additional analytical metrics needed to associate Level 2 objects to concepts and to compare concepts.

5.2 Level 3 fusion

The conceptual space approach can be applied to Level 3 fusion as well. The prediction of the intentions of an adversary can be approached by at least two different methods, both of which are amenable to operating on a conceptual space construct. The most straightforward of these is to define concepts pertaining to intentions, and then calculate the subsethood of these concepts in the Level 2 situational concepts determined from observations, as described in Sect. 4 above. This subsethood value provides a direct measure of the degree to which a prediction concept is matched by observations. This approach subsumes both rule-based and pattern-based predictions.

The other approach employs game theory to explore various courses of action, and makes its predictions based upon the most favorable strategies in which an adversary could engage. Here, the space of the game can be the con-
ceptual space, and the menu of candidate moves can be embodied as concepts. For example, the concept “attack position \((x, y)\)” specifies a set of kinematic, composition, formation and area of influence properties that a Level 2 object must possess in order to carry out such an attack. The degree to which an observation matches to this concept is thus predictive of possible enemy intentions.

6 Conclusion

In this paper, we extend Gardenfors’ notion of a “concept” in a conceptual space to be represented as a point in a unit hypercube that captures all of the concepts’ properties, salience weights and correlations. This description of a concept can be learned from a training data set. We also present a method for similarly describing a particular observation as a point in the same unit hypercube. We propose the mutual subsethood metric for measuring the similarity between two concepts, and the fuzzy subsethood metric for measuring the similarity of an observation to a concept. Both of these metrics are inherently normalized to values in \([0,1]\), which allows them to be ranked and thresholded without further normalization in order to declare situations and/or alert to novelties. Finally, we discuss the application of this technology to Level 2 and Level 3 data fusion problems.

References


